CONFORMAL MAPPING IN CALCULATING THE SOLIDIFICATION OF A COMPLEX CASTING

Yu. A. Samoilovich

UDC 536.42

An approximate method is given for calculating the solidification of a complex casting on the basis of a quasistationary approach and the use of conformal transformation. The solution to the two-dimensional Stefan's problem is illustrated by the solidification of a melt in wedges of angle up to 2π.

Profile castings and other such items have a great variety of geometrical forms; preference is often given to casting in preparing objects of complex shape, since one then often cannot use welding, forging, or stamping. On the other hand, existing methods of calculating casting solidification [1-8] allow one to calculate the motion of the solid front only for very simple geometrical forms such as plate, cylinder, or sphere. Exact solution of Stefan's problem (the basis of the solidification) is difficult and involves approximate methods, e.g., an integral balance relation, which can incorporate the enthalpy change in the solid crust and the heat released by the phase transition. This method was first used by Leibenzon in 1939 [9] for bodies of simple shape and provided simple working formulas for the thickness of the solid crust as a function of time.

We have derived comparatively simple working formulas as in [9] that define the shape of the solidification front and the speed for more complex castings; the initial assumptions are as follows:

a) the melt filling the mould is a metal or alloy with a very narrow crystallization range and such that the phase-transition temperature can be considered as constant at T_{cr};

b) we neglect any convection in the melt, whatever the cause (jet motion during pouring, nonuniform density, etc);

c) any superheating is neglected, i.e., we assume that the liquid is at the crystallization point throughout the process;

d) the thermophysical characteristics of the material (λ, ρ, and c_v) do not vary during the process;

e) we consider the two-dimensional temperature distribution and heat-flux pattern in the solid crust. The working regions in the cross section have the following features:

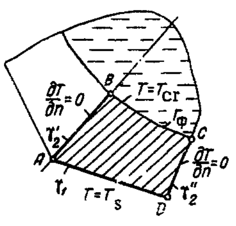

Fig. 1. Calculation region of canonical form.

1) a working region is two-dimensional and planar;

2) such a region is singly coupled;

3) the region is represented by the curvilinear rectangle ABCD (Fig. 1), in which the isothermal part of the solidification front BC is separated from the cooled surface AD by the adiabatic parts AB and CD. A working region that satisfies these three conditions is said to be a region of canonical form;

f) the temperature is assumed to be identical throughout the cooled part of the working region and unvarying (boundary condition of the first kind).

All-Union Metallurgical Heat-Engineering Scientific-Research Institute, Sverdlovsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 3, pp. 527-534, September, 1975. Original article submitted July 10, 1973.

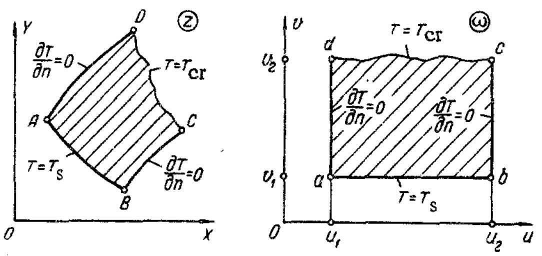

Fig. 2. Mapping of a calculation region of canonical form on a rectangle.

These simplifying assumptions concern the physical state of the melt and the geometrical features of the region, and they are accompanied by the three following postulates as to the solidification.

A. The isotherms for the solidification front remain of the same shape throughout the process (self-modelling postulate).

It is clear that this mode of solidification occurs when there are no sharp changes in the parameters of the cooling medium; an experimental confirmation of the postulate has been given ([10], p. 60) for castings solidifying in wedges.

B. The law of conservation of energy for a working region of canonical form is represented by

$$\int_0^l \left(\lambda \frac{\partial T}{\partial n}\right)_{\gamma_1} dl = \rho\,[L + c_r (T_{cr} - \bar{T})]\,\frac{dS\,(t)}{dt}\,. \tag{1}$$

The left side of this equation represents the heat consumption (per unit layer thickness) due to the cooled part of the casting γ_1 of length l, while the right side incorporates the heat released by the phase transition and the enthalpy change in the solid crust as the temperature falls from T_{cr} to $\bar{T}$, with $\bar{T}(t) = 1/S\iint T(x, y, t)dxdy$ the mean-mass temperature of the solid crust at time t and $S(t) = \iint_{(S)} dxdy$ the area of cross section of the solid crust at time t.

C. The temperature distribution in the working region of canonical form is determined by the solution for the corresponding problem in stationary heat conduction.

The latter postulate is an expression of the quasistationary approach to this case.

Analysis indicates that the error in determining the solidification time from the quasistationary approach does not exceed 8-10%, which is quite acceptable for engineering purposes.

The quasistationary postulate allows us to perform conformal mapping of the working region of canonical form and thus to produce an approximate solution.

We now show how conformal mapping can be used to calculate the solidification of a complex casting.

We introduced two planes of the complex variable $Z = X + iY$ and $\omega = u + iv$; in the Z plane we select a working region d of canonical form bounded by isotherms ($T = T_{cr}$, $T = T_s$) and current lines forming a mutually orthogonal net; the analytical function

$$\omega = u + iv = f\,(Z) \tag{2}$$

is used to map region d on region D (ω plane), the latter being of simpler form, on the basis that the solution to the stationary heat-conduction problem for region D is known and has a simple form. We assume that the function $\omega = f(Z)$ coincides apart from the real constants A and B with the complex thermal potential W(Z), i.e.,

$$W\,(Z) = A\omega\,(Z) + B = Q + i\theta, \tag{3}$$

from which we conclude that the straight lines v = const and u = const are the mapping of the isotherms and current lines on the ω plane. In the example considered below, the working region ABCD of canonical form is mapped into the band (rectangle) abcd, whose boundaries ab and cd are isotherms, while the boundaries bc and da are adiabatic current lines (Fig. 2). The stationary temperature distribution in the band abcd is one-dimensional and linear:

$$T = T\,(v) = T_s + (T_{cr} - T_s)\,\frac{v - v_1}{v_2 - v_1}\,, \tag{4}$$

and so

$$\frac{\partial T}{\partial v} = \frac{T_{cr} - T_s}{v_2 - v_1}\,,\quad \bar{T} = \frac{1}{2}\,(T_{cr} + T_s). \tag{5}$$

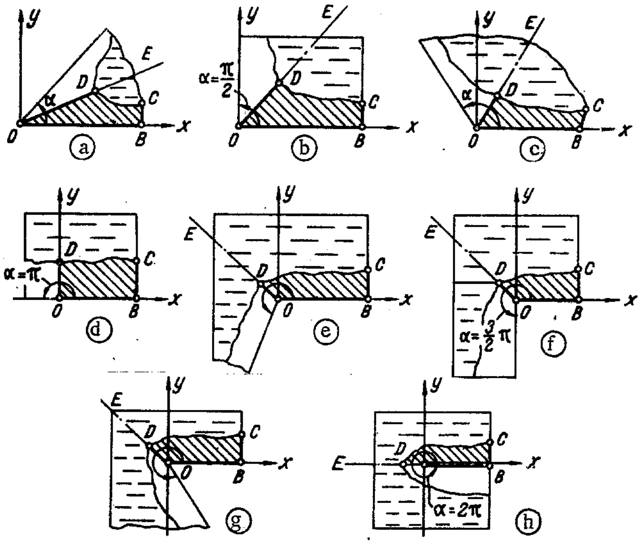

Fig. 3. Selection of calculation region for a corner in a casting.

On transferring from the variables x and y to the new variables u and v the equation for the heat balance of (1) becomes

$$\int_{u_1}^{u_2} \lambda \left(\frac{\partial T}{\partial v}\right)_{v=v_1} du = \rho\,[L + c_v\,(T_{cr} - \bar{T})]\,\frac{dS(t)}{dt} \tag{6}$$

or from (5)

$$\lambda\,(T_{cr} - T_s)\,\frac{u_2 - u_1}{v_2 - v_1} = \rho\left[L + \frac{c_v}{2}\,(T_{cr} - T_s)\right]\frac{dS(t)}{dt}\,. \tag{7}$$

The area of cross section of the casting at time t is

$$S(t) = \iint\limits_{(S)} dxdy = l^2 \iint\limits_{(S)} dXdY,$$

or on transferring to the variables u and v,

$$S(t) = l^2 \iint dXdY = l^2 \iint D(u,\ v)\,dudv, \tag{8}$$

where D(u, v) is a functional determinant (Jacobian), whose expression takes the following form ([11], p. 246):

$$D(u,\ v) = \frac{\partial X}{\partial u}\,\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v}\,\frac{\partial Y}{\partial u}\,. \tag{9}$$

The relationships

$$X = X(u,\ v),\ Y = Y(u,\ v) \tag{10}$$

can be found by inverting the mapping function of (2), i.e.,

$$Z = X + iY = f_1(\omega), \tag{11}$$

and by comparing the real and imaginary parts of the latter equation. Since

$$\frac{dS}{dt} = \frac{\partial S}{\partial \xi}\,\frac{d\xi(t)}{dt}\,, \tag{12}$$

where $\xi(t) = v_2(t)$ is the value of the coordinate v = const corresponding to the position of the solidification front at time t, we have

$$\frac{dS}{dt} = l^2\,\frac{d\xi}{dt}\int_{u_1}^{u_2} D(u,\ \xi)\,du \tag{13}$$

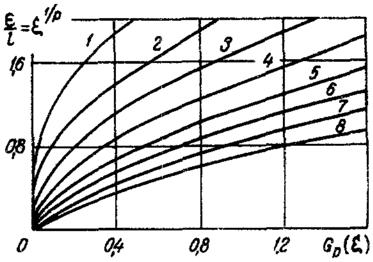

Fig. 4. Melt solidification at corners with angles α that are multiples of $\pi/4$: 1) $\pi/4$; 2) $\pi/2$; 3) $3\pi/4$; 4) π; 5) $5\pi/4$; 6) $3\pi/2$; 7) $7\pi/4$; 8) 2π, $[2K\theta_s/(2+K\theta_s)]\tau = G_p(\xi)$.

and (7) becomes

$$\lambda(T_{cr}-T_s)\frac{u_2-u_1}{v_2-v_1}dt = \rho l^2\left[L+\frac{c_v}{2}(T_{cr}-T_s)\right]d\xi\int_{u_1}^{u_2}D\,du. \tag{14}$$

We assume for simplicity that $u_1 = 0$, $v_1 = 0$, $u_2 = 1$, $v_2 = \xi$, and then the latter equation takes the form

$$\frac{\lambda}{\rho c_v}\,\frac{c_v(T_{cr}-T_s)}{L}\,\frac{dt}{l^2\left[1+\frac{c_v(T_{cr}-T_s)}{2L}\right]} = \xi d\xi\int_0^1 D(u,\ \xi)\,du. \tag{15}$$

We introduce the dimensionless complexes

$$\frac{\lambda}{\rho c_v}\,\frac{t}{l^2}=\tau,\quad \frac{c_v(T_{cr}-T_m)}{L}=K,\quad \theta_n=\frac{T_{cr}-T_s}{T_{cr}-T_m}$$

and integrate (15) subject to the initial condition

$$\xi = 0 \ \text{ for } \ t = 0, \tag{16}$$

to get the solution in the form

$$\frac{2K\theta_s}{2+K\theta_s}\tau = \int_0^{\xi}\xi d\xi\int_0^1 D(u,\ \xi)\,du, \tag{17}$$

where the geometrical features of the working region are incorporated in D.

As an example of the use of (17) we consider the solidification of a melt at a corner (angles up to 2π); Fig. 3 shows the various possible styles.

The working region OBCD (Fig. 3a) is bounded by the cool part OB, the isotherm CD, and the two adiabatic current lines BC and OD. We introduce the parameter $p = \pi/a$. The working region is represented by α between 0 and 360°, which corresponds to values of $p = \pi/\alpha$ from 1/2 to infinity, and in particular in the first quadrant $(\pi/2 \geq \alpha > 0)\ 2 \leq p < \infty$, while in the second quadrant $(\pi \geq \alpha \geq \pi/2)\ 1 \leq p \leq 2$, in the third quadrant $(3\pi/2 \geq \alpha \geq \pi)\ 2/3 \leq p \leq 1$, and in the fourth quadrant $(2\pi > \alpha \geq 3\pi/2)\ 1/2 < p \leq 2/3$.

The analytical function

$$\omega = u + iv = Z^p \tag{18}$$

represents conformally the working region OBCD in the band obcd, and the Jacobian is

$$D = \frac{1}{p^2}(u^2+v^2)^{(1/p)-1}. \tag{19}$$

We use the general solution of (17) to find from (19) that

$$\frac{2K\theta_s}{2+K\theta_s}\tau = \frac{1}{2p}\int_0^1 (u^2+\xi^2)^{\frac{1}{p}}du - \frac{1}{2(p+2)}. \tag{20}$$

The thickness of the layer of solid crust ε in the direction of the bisector of α is related to ξ by

$$\varepsilon(t) = l\xi^{1/p}(\tau), \tag{21}$$

which follows from the correspondence between the points in D and d:

$$r_D = \frac{\varepsilon}{l},\ \theta_D = \frac{\alpha}{2} = \frac{\pi}{2p},\ u_d = 0,\ v_d = \xi = r_D^p \sin p\theta = \left(\frac{\varepsilon}{l}\right)^p \sin\frac{\pi}{2} = \left(\frac{\varepsilon}{l}\right)^p.$$

For $\alpha = 90°$ (Fig. 3b) the time dependence of ξ takes the form

$$\frac{2K\theta_s}{2+K\theta_s}\tau = \frac{1}{8}\left(\xi^2\ln\frac{1+\sqrt{1+\xi^2}}{\xi}+\sqrt{1+\xi^2}-1\right). \tag{22}$$

Parameter ξ is related to ε along OE by

$$\varepsilon(t) = l\sqrt{\xi(t)}, \tag{23}$$

which follows from the correspondence between the points in D and d:

$$X_D = Y_D = \frac{\varepsilon}{l}\cos\frac{\pi}{4},\quad u_d = 0,\quad v_d = \xi = 2\left(\frac{\varepsilon}{l}\cos\frac{\pi}{4}\right)^2 = \left(\frac{\varepsilon}{l}\right)^2.$$

For $\alpha = 270°$ (for $p = 2/3$) we have (Fig. 3f):

$$\varepsilon(t) = l\xi^{3/2}(\tau).$$

For this case the solution of (20) is taken in quadratures and takes the form

$$\frac{2K\theta_s}{2+K\theta_s}\tau = \frac{9}{8}\left[-\frac{\xi^4}{4}\ln\frac{1+\sqrt{1+\xi^2}}{\xi} - \frac{1}{6} + \frac{2+5\xi^2}{12}(1+\xi^2)^{1/2}\right]. \tag{24}$$

The form of the solidification front at an interior right angle (for $\alpha = 270°$) is determined from the condition $v = \xi(t)$ subject to the expressions

$$\xi = r^{2/3}\sin\frac{2}{3}\theta = \text{const},\ r = (X^2+Y^2)^{1/2},\ \theta = \operatorname{arctg}\frac{Y}{X},$$

whence

$$\xi(t) = (X^2+Y^2)^{1/3}\sin\left(\frac{2}{3}\operatorname{arctg}\frac{Y}{X}\right).$$

The integral in (20) can be found numerically for arbitrary values of α.

Figure 4 shows results for angles that are multiples of $\pi/4$, i.e., for $\alpha = \pi/4, \pi/2, 4\pi/3$, etc.

The ratio $\varepsilon_\alpha/\varepsilon_c$ gives an indication of how α influences the speed of the front, where ε_c is the thickness of the crust for $\alpha = 180°$, i.e., under the conditions of Stefan's problem. This method of incorporating the effects of the angle between conjugate planes has been used [10] in experiments on solidification.

Note that within the range of variation in $K\theta_s$ and τ of practical interest for metallic castings, the ratio $\varepsilon_\alpha/\varepsilon_c$ is not a constant (Fig. 4), but the limits of variation are not too large and are as follows: $\varepsilon_{45}/\varepsilon_c$ = 2.2-2.7 for $\alpha = 45°$; $\varepsilon_{90}/\varepsilon_c$ = 1.5-1.8 for $\alpha = 90°$; $\varepsilon_{135}/\varepsilon_c$ = 1.2-1.35 for $\alpha = 135°$; $\varepsilon_{225}/\varepsilon_c$ = 0.76-0.84 for $\alpha = 225°$; $\varepsilon_{270}/\varepsilon_c$ = 0.66-0.74 for $\alpha = 270°$; $\varepsilon_{315}/\varepsilon_c$ = 0.50-0.64 for $\alpha = 315°$; $\varepsilon_{360}/\varepsilon_c$ = 0.40-0.53 for $\alpha = 360°$.

For comparison, we give measured values from the data of [10] (p. 60): $\varepsilon_{45}/\varepsilon_c$ = 2.9-3.1; $\varepsilon_{135}/\varepsilon_c$ = 1.25-1.4; $\varepsilon_{90}/\varepsilon_c$ = 1.6-1.9; $\varepsilon_{270}/\varepsilon_c$ = 0.48-0.7.

It may be seen that the calculated $\varepsilon_\alpha/\varepsilon_c$ agree satisfactorily with the experimental values due to Gulyaev.

NOTATION

T	is the temperature;
T_{cr}, T_m	are the crystallization temperature and temperature of cooling medium, respectively;
t	is the time;
x, y	are the coordinates;
λ, ρ, c_v, L	are the thermal conductivity, density, specific heat, and latent heat of fusion, respectively;
l	is the characteristic linear dimension;
Q_1	is the heat flow rate per unit time per unit bed thickness;
$i = \sqrt{-1}$, $i^2 = -1$; $Q = Q_1/\lambda(T_{cr}-T_m)$	is the dimensionless heat flow rate;
$\theta = (T-T_m)/(T_{cr}-T_m)$	is the dimensionless temperature;
$X = x/l$, $Y = y/l$	are the dimensionless coordinates;
$K = c_v(T_{cr}-T_m)/L$	is the thermophysical criterion;
$\tau = (\lambda/\rho c_v)(t/l^2)$	is the Fourier number.

LITERATURE CITED

1. B. Ya. Lyubov, Dokl. Akad. Nauk SSSR, 68, No. 5 (1949).
2. B. Ya. Lyubov, Dokl. Akad. Nauk SSSR, 92, No. 4 (1953).
3. V. T. Borisov, B. Ya. Lyubov, and D. E. Temkin, Dokl. Akad. Nauk SSSR, 104, No. 2 (1955).
4. J. Kolodner, in: Mechanics [Periodic collection of translations of foreign articles], No. 1 (41) (1957).
5. G. A. Tirskii, Dokl. Akad. Nauk SSSR, 125, No. 2 (1959).

6. G. P. Ivantsov, B. Ya. Lyubov, B. G. Roitburd, Inzh.-Fiz. Zh., 3, No. 3 (1960).
7. E. K. Iodko, Inzh.-Fiz. Zh., 6, No. 7 (1963).
8. S. M. Adams, in: Solidification of Liquid Metals [Russian translation], Metallurgizdat, Moscow (1962).
9. L. S. Leibenzon, Collected Works [in Russian], Vol. 4, Izd. Akad. Nauk SSSR, Moscow (1955).
10. B. B. Gulyaev, Solidification and Inhomogeneity in Steel [in Russian], Metallurgizdat, Moscow–Leningrad (1950).
11. V. I. Smirnov, Textbook of Higher Mathematics [in Russian], Vol. 2, Nauka, Moscow (1965).